5 S. Vagi, Harmonic analysis on Cartan and Siegel domains, in *Studies in Harmonic Analysis* (J. M. Ash, Ed.), Math. Assoc. Amer., 1976.
6 J. A. Wolf, Fine structure of Hermitian symmetric spaces, in *Symmetric Spaces* (W. Boothy and G. Weiss, Eds.), Marcel Dekker, New York, 1972, pp. 271–357.

Hermitian Forms, Full Rings, and von Neumann Regular Matrices

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At the last Dublin matrix conference (October 1980) and in more detail in [3] we reviewed, from a matrix viewpoint, the isometry classification problems for quadratic and Hermitian forms over fields and division algebras. Here we discuss these kinds of forms over more general rings. We do not attempt any kind of survey, but simply mention a couple of approaches of potential interest to matrix theorists.

1. *Full Rings*

Motivated by the matrix-theoretic proof of the diagonalizability of quadratic forms over fields, McDonald and Hershberger [4] define the notion of a *full ring* (a ring which is full of units in a certain sense). Their definition goes as follows: Let R be a commutative ring in which 2 is a unit. Then R is said to full if it is n-full for each positive integer n, where n-full means that given $2n+1$ elements $\alpha, \beta_1, \dots, \beta_n, \delta_1, \dots, \delta_n$ of R, such that the ideal generated by these elements is all of R, there exist $\omega_1, \dots, \omega_n$ in R such that $\alpha + \sum_{i=1}^n \beta_i \omega_i + \sum_{i=1}^n \delta_i \omega_i^2$ is a unit of R. They show that many familiar classes of ring satisfy this. In particular, local and semilocal rings, von Neumann regular rings, and zero-dimensional rings are all full.

They prove that any nonsingular symmetric bilinear form $\phi: M \times M \to R$ (M being a free R-module of rank n and nonsingular meaning that the map $M \to \operatorname{Hom}_R(M, R)$, $m \to h(m, \cdot)$ is bijective) has a diagonal matrix representation. In matrix terms, any nonsingular symmetric $n \times n$ matrix with entries in R is congruent to a diagonal matrix, i.e., there exists an invertible $n \times n$ matrix P such that $P^t A P$ is diagonal. The idea of the proof is to do exactly as in the field or skewfield case; see [3, pp. 249–250]. As long as the top left-hand corner entry of A is a unit, then a congruence transformation and an induction argument complete the proof. In order to obtain this, the proof in [4] starts by using the fact that the form is nonsingular and the definition of full ring to make the (1,2) entry of A into a unit via a congruence transformation. Then, another congruence transformation, again using the definition of full, leads to a matrix whose (1,1) entry is a unit.

It would seem that it is possible to extend the definition of full ring to the situation of a not necessarily commutative ring with involution. (By an involution we mean an antiautomorphism of period less than or equal to two.) We suggest the following definition: Let R be a ring with involution in which 2 is a unit. R is *full* if it is n-full for each positive integer n, where n-full means that given any $2n+1$ elements $\alpha, \beta_1, \dots, \beta_n, \delta_1, \dots, \delta_n$ of R such that these elements together with their image under the involution generate all of R, there exist elements $\omega_1, \omega_2, \dots, \omega_n$ in R such that $\alpha + \bar{\alpha} + \sum_{i=1}^{n}(\beta_i\omega_i + \overline{\beta_i\omega_i}) + \sum_{i=1}^{n}\bar{\omega}_i\delta_i\omega_i$ is a unit (- denotes the involution). The diagonalization proof goes through in a similar way to the above, except that this time we make the (1,2) entry to be b_{12} where $b_{12} + \bar{b}_{12}$ is a unit, instead of b_{12} itself being a unit. In matrix terms diagonalization means of course that for nonsingular A with $A = \bar{A}^t$ there exists P with $\bar{P}^tAP$ diagonal.

In [4], where McDonald and Hershberger deal exclusively with commutative rings, the definition of full enables a lot of the basic theory of nonsingular symmetric bilinear forms and their orthogonal groups to be developed, e.g. a Witt cancellation theorem, generators for the orthogonal group, etc. The same authors, in [5], go on to describe the Witt ring of nonsingular symmetric bilinear forms over a full ring in the commutative case. Under our proposed definition of full for noncommutative rings with involution, we have shown that diagonalizability of forms works. It is possible that a lot of the other results of [4, 5] may go through also, but we have not examined that. Another question to consider would be whether or not any particular classes of noncommutative rings with involution satisfy our definition. For example, are noncommutative von Neumann regular rings with involution full under our definition?

2. *Von Neumann Regular Matrices*

Let R be a ring with involution (R not necessarily commutative). Consider Hermitian forms $h: M \times M \to R$, M being an R-module. Kanzaki [1] adopts a matrix approach to such forms by taking a generating set $\{m_i\}$ for M (possibly infinite) and forming the matrix $h(m_i, m_j)$. The form h is nonsingular if $M \to \operatorname{Hom}_R(M, R)$, $m \to h(m, \cdot)$ is bijective. If h is nonsingular and $\{m_i\}$ is a finite set with n elements, then $H = (h(m_i, m_j))$ is an $n \times n$ Hermitian matrix, and it turns out that M is isomorphic to R^nH, R^n being n-tuples of elements of R written as rows. [The isomorphism is given by $M \to R^nH$, $\sum a_i m_i \to (a_i)H$. The nonsingularity ensures that this is a well-defined isomorphism.]

If, and only if, M is projective, Kanzaki shows that H must be von Neumann regular in the sense that there exists an $n \times n$ matrix K with $HKH = H$. (For skewfields, and indeed semisimple rings, all matrices are von Neumann regular, and so it is only for more general rings that this has much significance.)

Given an $n \times n$ Hermitian von Neumann regular matrix H, then there is a well-defined nonsingular Hermitian form $R^nH \times R^nH \to R$, $((a_i)H, (b_i)H) \to \sum\sum a_i h_{ij} \bar{b}_j$ [where $H = (h_{ij})$], R^nH being a projective R-module. Denote this form by $\langle H \rangle$. Kanzaki shows that every finitely generated projective nonsingular Hermitian R-module is isometric to $\langle H \rangle$ for some finite von Neumann regular matrix H. The isometry problem may be formulated in matrix terms as follows: $\langle H_1 \rangle$ and $\langle H_2 \rangle$ are isometric if there exist matrices L, L' of size $n_1 \times n_2$, $n_2 \times n_1$ respectively (where H_i is $n_i \times n_i$, $i = 1,2$), such that

$$LH_2\bar{L}^t = H_1, \qquad LL'H_1 = H_1, \qquad L'LH_2 = H_2.$$

(Note that the first equation is the usual congruence one; the other two are needed to induce isomorphisms between $R^{n_1}H_1$ and $R^{n_2}H_2$.) This matrix problem does not seem all that tractable.

Also in [1], matrix criteria for $\langle H \rangle$ to be hyperbolic and metabolic are given, and in addition a matrix description of the orthogonal group of $\langle H \rangle$ is presented.

An application of the methods of [1] appears in [2].

REFERENCES

1 T. Kanzaki, Notes on hermitian forms over a ring, *J. Math. Soc. Japan* 30:723–735 (1978).

2 T. Kanzaki, A note on Witt groups for a Frobenius extension with involution, *J. Pure Appl. Algebra* 22:249–252 (1981).

3 D. W. Lewis, The isometry classification of hermitian forms over division algebras (Report), *Linear Algebra Appl.* 43:245–272 (1982).

4 B. McDonald and B. Hershberger (Kirkwood), The orthogonal group over a full ring, *J. Algebra* 51:536–549 (1978).

5 B. Kirkwood and B. McDonald, The Witt ring of a full ring, *J. Algebra* 64:148–166 (1980).

Grassmann Matrices

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